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Orlicz spaces cannot be normed analogously to L^p-spaces

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ABSTRACT

Using the multipliers of an Orlicz function F we show that the functional

$$p_{G,F}(x) = G\Big(\int_S F(|x(s)|)d\mu\Big),$$

defined on the Orlicz space $L^F(\mu)$, is homogeneous if and only if $F(r)=F(1)r^p$ and $G(r)=G(1)r^{1/p}$ for some $p>0$ and all $r\geq 0$. This statement is proved under very general assumptions about the measure μ.

The historical survey [4] contains an elegant and elementary proof of the fact that the functional p_F, defined on the Orlicz space L^F (over the Lebesgue measure on the real line) by the formula

$$p_F(x) = F^{-1}\Big(\int_{\mathbb{R}} F(|x(s)|)ds\Big),$$

is homogeneous iff F is a power function. To ensure the correctness of the definition of p_F it is assumed that F satisfies a Δ_2-condition. If $F(r)=cr^p$ ($p\geq 1$, $c>0$), then p_F is of course the usual L^p-norm on $L^F=L^p$. The above equivalence thus shows that the natural analogue of the L^p-norm is of no use in the case of general Orlicz spaces. The relation between the homogenity of p_F and the form of F has been known for almost fifty years – Professor Zaanen mentions in [4] a relevant remark in Antoni Zygmund's book *Trigonometrical series* edited in 1935. But, as far as we know, no proof of that equivalence has been published before Zaanen's paper [4]. However, the proof has one "inconvenience" – it requires the existence of a sequence of disjoint sets of measure one.

The main purpose of our paper is to extend the result proved by Zaanen to functionals of the form

$$p_{G,F}(x) = G\Big(\int_S F(|x(s)|)d\mu\Big),$$

where the measure μ in S can be taken almost arbitrarily (some pathological cases are only excluded).

Let F be an *Orlicz function*, i.e., $F\colon [0,\infty)\to[0,\infty)$ is unbounded, strictly increasing, continuous and $F(0)=0$. A number $m>0$ is called a *multiplier* of F if $F(mr)F(1)=F(m)F(r)$ for all $r\geq 0$. The set $\mathcal{M}_F$ of all multipliers of F is a closed subgroup of the multiplicative group of strictly positive real numbers. Therefore $\mathcal{M}_F=(0,\infty)$ or $\mathcal{M}_F$ is cyclic, i.e., $\mathcal{M}_F=\{d^k\colon k\in\mathbb{Z}\}$ for some $d>0$, where $\mathbb{Z}$ denotes the set of integers.

We refer the reader interested in multipliers to [1], [2], [3]. We assume familiarity with the definition and basic properties of the Orlicz space $L^F(\mu)$ and its subspace

$$L_a^F(\mu)=\{x\in L^F(\mu)\colon \int_S F(r|x(s)|)d\mu<\infty \text{ for all } r>0\}.$$

Let us only recall that if F satisfies a suitable Δ_2-condition (whose form depends on the measure μ), then $L^F(\mu)=L_a^F(\mu)$.

The symbols 1_A, F^{-1} are reserved for the characteristic function of a set A and for the inverse function of F.

THEOREM. *Let μ be a measure such that there exist two disjoint sets of finite and positive measure. If F and G are Orlicz functions, then the functional $p_{G,F}$ defined on $L_a^F(\mu)$ is homogeneous iff $F(r)=F(1)r^p$ and $G(r)=G(1)r^{1/p}$ for some $p>0$.*

PROOF. Let A,B be disjoint sets such that $a=\mu(A)$ and $b=\mu(B)$ are positive and finite. Assume $p_{G,F}$ is homogeneous. Thus

$$G(F(r)a)=p_{G,F}(r1_A)=rp_{G,F}(1_A)=rG(F(1)a)$$

holds for $r\geq 0$. Writing $w=G(F(1)a)$, we have

$$G^{-1}(r)=aF(r/w) \text{ and } G(r)=wF^{-1}(r/a).$$

For every set C of positive finite measure we have $1_C\in L_a^F(\mu)$, and for all r,

$$wF^{-1}(F(r)\cdot\mu(C)/a)=p_{G,F}(r1_C)=rp_{G,F}(1_C)=rwF^{-1}(F(1)\cdot\mu(C)/a).$$

Putting $m_C=F^{-1}(F(1)\cdot\mu(C)/a)$ we obtain $m_C\in\mathcal{M}_F$. In particular, $F^{-1}(F(1)b/a)\in\mathcal{M}_F$.

We claim $\mathcal{M}_F=(0,\infty)$. Indeed, suppose $\mathcal{M}_F=\{d^k:k\in\mathbb{Z}\}$. We can assume $d\geq 1$. Let $x_{n,m}=d^n 1_A+d^m 1_B$, where $n,m\in\mathbb{Z}$. For every $r\geq 0$ we have

$$\begin{aligned} &wF^{-1}((F(r)/F(1))(F(d^n)a+F(d^m)b)/a)=wF^{-1}((F(rd^n)a+F(rd^m)b)/a)\\ &=p_{G,F}(rx_{n,m})=rp_{G,F}(x_{n,m})=rwF^{-1}((F(d^n)a+F(d^m)b)/a). \end{aligned}$$

Writing $v_{n,m}=F^{-1}((F(d^n)a+F(d^m)b)/a)$, we can easily see that $v_{n,m}\in\mathcal{M}_F$. Hence

$$F(d^n)+F(d^m)\frac{b}{a}=F(d^j)$$

for some integer j. We have already proved that $F^{-1}(F(1)b/a)\in\mathcal{M}_F$ and therefore $b/a=F(d^i)/F(1)$ for some integer i. Thus

$$F(d^n)+F(d^{m+i})=F(d^j).$$

It follows that $d>1$ and $n+1\leq j$. Now specify $n>0$ and m so that $F(d^{m+i})<$ $<F(d)-F(1)$ (such a choice of m is possible because F is continuous). Using induction we prove $F(d^k)=F(d)^k/F(1)^{k-1}$ for all integers k. Therefore, putting $s=F(d)/F(1)$, we get

$$s^n+F(d^{m+i})/F(1)=s^j.$$

Thus $s^j<s^n+(s-1)<s^n+s^n(s-1)=s^{n+1}$. Since $s>1$, we must have $j<n+1$; a contradiction.

Finally $\mathcal{M}_F=(0,\infty)$, and so $F(r)=F(1)r^p$ for some $p>0$. We also obtain $G(r)=G(1)r^{1/p}$ from the equality $G(r)=wF^{-1}(r/a)$, proved at the beginning.

REMARKS. The existence of two disjoint sets of positive finite measure is equivalent to the condition dim $L^F(\mu)>1$.

The same proof shows that $p_{G,F}$ is q-homogeneous, i.e., $p_{G,F}(rx)=r^q p_{G,F}(x)$, iff $F(r)=F(1)r^p$ and $G(r)=G(1)r^{q/p}$ for some $p>0$.

PROBLEM. Describe those pairs of G,F for which $p_{G,F}$ is subadditive (then $p_{G,F}$ will be an F-norm).

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